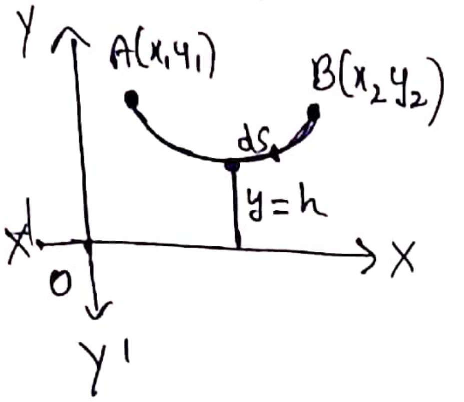
Hanging chain problem:

A heavy cable or chain hangs freely under gravity between two fixed points. Show that the shape of the cable is a catenary.



Let a chain or cable hangs freely betⁿ points $A(x_1, y_1)$ and $B(x_2 y_2)$.

Let m be mass of the element of length ds. and ρ be the density. Then $m = \rho ds$.

The potential energy of the element $= mgh = (\rho ds) g h = \rho ds g y$

$\therefore$ Total potential energy of the cable $= T = \int_A^B \rho ds g y = \int_A^B \rho g y ds$

ie $T = \int_{x_1}^{x_2} \rho g y \cdot \frac{ds}{dx} dx$ $\qquad \frac{ds}{dx} = \sqrt{1+y'^2}$

$$T = \rho g \int_{x_1}^{x_2} y \sqrt{1+y'^2}\, dx$$

$f = y\sqrt{1+y'^2}$. Eulers Eqⁿ $f - y' \frac{\partial f}{\partial y'} = k$

$$y\sqrt{1+y'^2} - y'\left(y \cdot \frac{y'}{\sqrt{1+y'^2}}\right) = k$$

$$y(1+y'^2) - y y'^2 = k\sqrt{1+y'^2}$$

$$y + \cancel{yy'^2} - \cancel{yy'^2} = k\sqrt{1+y'^2}$$

i.e $y^2 = k^2(1+y'^2)$

$$y'^2 = \frac{y^2}{k^2} - 1 = \frac{y^2-k^2}{k^2}$$

$$y' = \frac{\sqrt{y^2-k^2}}{k}$$

$$\frac{k}{\sqrt{y^2-k^2}}\, dy = dx + k_2$$

$$\cosh^{-1}\frac{y}{k} = \frac{x}{k} + k_2.$$

$$\cosh^{-1}\frac{y}{k} = \frac{x}{k} + \frac{k_2}{\cancel{k}}$$

$$\cosh^{-1}\frac{y}{k} = \frac{x}{k} + K_2$$

$$\frac{y}{k} = \cosh\left(\frac{x}{k} + k_2\right)$$

$$y = k\cosh\left(\frac{x + kk_2}{k}\right)$$

Put $k = c$. $kk_2 = a$

$$y = c\cosh\left(\frac{x+a}{c}\right)$$

This equation represents a catenary

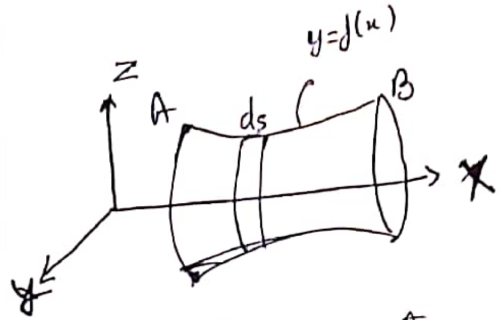
— Minimal Surface of revolution :

Prove that Catenary is the curve which when rotated about a line generates a Surface of minimum area.

Let $A(x_1, y_1)$ & $B(x_2, y_2)$ be two fixed points. Let $y = f(x)$ be the curve passing between these fixed end points.

[When the curve $y = f(x)$ revolves around x-axis it generates a surface.]

Now let us consider the surface generated by revolving the curve $y = f(x)$ which is called Surface of revolution.



equation of catenery
$y = c\cosh\left(\frac{x+a}{c}\right)$ $a = Kc$

The problem is to find the curve for which surfacearea is minimum.

$$\text{Total Surface area} = \int_{x_1}^{x_2} 2\pi y \, ds \quad -① \quad (\text{formula})$$

here $ds = \sqrt{1+y'^2}\, dx$.

$$S = \int_{x_1}^{x_2} 2\pi y \sqrt{1+y'^2}\, dx = 2\pi \int_{x_1}^{x_2} y\sqrt{1+y'^2}\, dx.$$

We have to find the curve y which minimizes S.
This is a variational problem.

$f = y\cdot\sqrt{1+y'^2}$. The function f does not contain x

$\therefore$ Euler's equation is $f - y'\dfrac{\partial f}{\partial y'} = c$ —— (1)

$f = y\sqrt{1+y'^2}$

$$\frac{\partial f}{\partial y'} = y\cdot\frac{\partial\sqrt{1+y'^2}}{\partial y'} = y\cdot\frac{1}{2\sqrt{1+y'^2}}\cdot\frac{\partial y'^2}{\partial y'} = y\cdot\frac{1}{\cancel{2}\sqrt{1+y'^2}}\cdot\cancel{2}y'$$

$$\frac{\partial f}{\partial y'} = \frac{yy'}{\sqrt{1+y'^2}}$$

Substitute in (1), $y\sqrt{1+y'^2} - y'\left[\dfrac{yy'}{\sqrt{1+y'^2}}\right] = c$

Take L.C.M. as $\sqrt{1+y'^2}$

$$\frac{y(1+y'^2) - y'^2 y}{\sqrt{1+y'^2}} = c$$

ie $y(1+y'^2) - y'^2 y = c\sqrt{1+y'^2}$

$y + \cancel{yy'^2} - \cancel{yy'^2} = c\sqrt{1+y'^2}$

$y = c\sqrt{1+y'^2}$

Squaring

$y^2 = c^2(1+y'^2)$

$$y'^2 = \frac{y^2}{c^2} - 1 = \frac{y^2-c^2}{c^2}.$$

$$y' = \frac{\sqrt{y^2-c^2}}{c}$$

$$\frac{dy}{dx} = \frac{\sqrt{y^2-c^2}}{c}$$

$$\frac{dy}{\sqrt{y^2-c^2}} = \frac{dx}{c}$$

integrate

$$\int\frac{dy}{\sqrt{y^2-c^2}} = \int\frac{dx}{c} + k.$$

$$\cosh^{-1}\frac{y}{c} = \frac{x}{c} + k$$

$$\frac{y}{c} = \cosh\left(\frac{x+kc}{c}\right)$$

$$\boxed{y = c.\cosh\left(\frac{x+a}{c}\right)},\ a = kc$$

This is the equation of Catenary

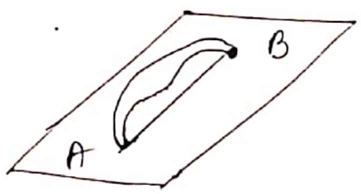
This curve generates ~~min~~ surface with minimum surface area

Surface ~~SA~~ is called catenoid

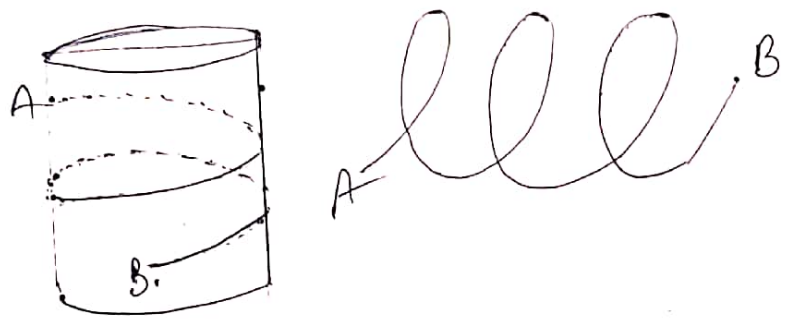
Geodesics :

A geodesic on a surface is a curve along which the length ~~of~~ the curve between two points on the surface is a minimum.

ex 1. Consider a plane. Let A and B be two points. Then A and B can be joined by many curves. These curves would have different lengths. The straight line has the minimum length between two points on a plane. Then geodesic on a plane is said to be st line.

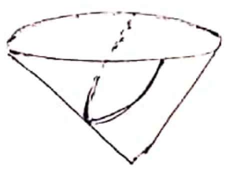


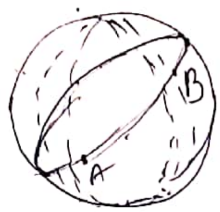
ex 2 : Geodesic on a right circular cylinder is a helix.



ex 3: Geodesic on a sphere is a curve on the great circle passing through two point A & B.

ex 4: Geodesic on a right circular cone is spiral.



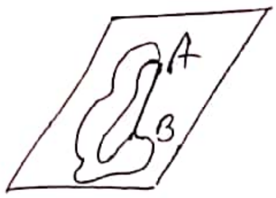


- Show that the shortest distance between two points in a plane is along a straight line joining them

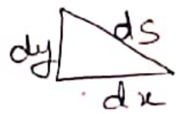
OR

Prove that the geodesics on a plane are straight ~~lines~~ lines.

Consider a plane XOY.
Let $y = y(x)$ be a curve joining two points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the plane XOY.



element of arc length $= ds$.



$$ds = \sqrt{dx^2 + dy^2}$$

$$\text{Total length} = S = \int_{x_1}^{x_2} ds$$

$$= \int_{x_1}^{x_2} \frac{ds}{dx}\, dx = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$S = \int_{x_1}^{x_2} \sqrt{1 + (y')^2}\, dx \quad \text{—— (1)}$$

To find y so that S is minimum. ie we have to find the curve $y = y(x)$ whose length S is minimum.
This is a variational problem.
S is a functional whose value is minimum
$\therefore f = \sqrt{1 + (y')^2}$ satisfies Euler's equation.
it does not contain x and y.
$\therefore$ Euler's Equation is $\boxed{y'' = 0}$

$$y'' = 0$$

Solve this differential equation

$$\int y''\,dx = k \qquad\qquad \int y'' = \int \frac{d^2y}{dx^2}\,dx = \frac{dy}{dx}$$

$$y' = k. \qquad\qquad \int \frac{dy}{dx}\,dx = y$$

$$\int y' = \int k + k_1$$

$$y = kx + k_1$$

i.e. $\boxed{y = ax + b}$

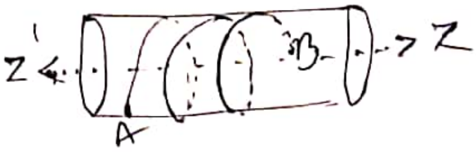
which is equation of a straight line.

and length is minimum.

$\therefore$ Hence geodesic on a plane are straight lines

7. Show that geodesic on a right circular cylinder is a helix. 4.

Consider a rt circular cylinder
Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points on it. Join A,B.



Let dl be element length.
Then arc length element in cylindrical polar coordinate (R, ϕ, z) is given by

$$ds = \sqrt{(dR)^2 + R^2(d\phi)^2 + (dz)^2}$$

Now arc lies on rt circular cylinder with axis as z-axis and radius of cross section is 'a'.

$$\therefore \quad R = a \Rightarrow dR = 0$$

$$ds = \sqrt{a^2 d\phi^2 + (dz)^2} = \sqrt{a^2 + \left(\frac{dz}{d\phi}\right)^2}\, d\phi$$

The total length of the curve is given by $= \int_{\phi_1}^{\phi_2} ds = \int_{\phi_1}^{\phi_2} \sqrt{a^2 + \left(\frac{dz}{d\phi}\right)^2}\, d\phi$

ϕ_1 and ϕ_2 are values of ϕ at A and B respectively

Then $$S = \int_{\phi_1}^{\phi_2} \sqrt{a^2 + \left(\frac{dz}{d\phi}\right)^2}\, d\phi \quad \text{—— (1)}$$

Let us take that S is minimum, then we need to find z for this minimum length.
S is a functional with minimum value $\therefore$ it satisfies Euler's equn. $f = \sqrt{a^2 + (z')^2}$
It does not contain ϕ and z

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$\therefore$ The ~~Eqn~~ Euler's Equation is

$$z'' = 0$$

integrate

$$\int z'' = C_1$$

$$\Rightarrow \quad z' = C_1$$

$$\int z' = \int c\,\cancel{d\phi} + C_2$$

ie
$$\int \frac{dz}{d\phi}\, d\phi = \int C_1\, d\phi + C_2$$

$$\int dz = \int C_1\, d\phi + C_2$$

$$\boxed{z = C_1 \phi + C_2} \quad -(2)$$

Equation (2) is Extremal of the functional (1)

ie Equation (2) is the equation of helix.

$\therefore$ helix is the geodesic of rt circular cylinder.

Geodesics

3. Find the geodesics on a surface given that the arc length on the surface is $S = \int_{x_1}^{x_2} \sqrt{x(1+y'^2)}\, dx$.

$f = \sqrt{x(1+y'^2)}$, f does not contain y

Euler's equation is $\boxed{\frac{\partial f}{\partial y'} = k}$ —— (1)

$$\frac{\partial f}{\partial y'} = \frac{\partial \sqrt{x(1+y'^2)}}{\partial y'} = \frac{1}{2\sqrt{x(1+y'^2)}} \cdot \frac{\partial x(1+y'^2)}{\partial x}$$

$$= \frac{1}{\cancel{2}\sqrt{x(1+y'^2)}} \times x \cdot \cancel{2} y' = \frac{xy'}{\sqrt{x(1+y'^2)}}$$

Sub in (1)

$$\frac{xy'}{\sqrt{x(1+y'^2)}} = k_1$$

$$\Rightarrow \quad xy' = k_1 \sqrt{x(1+y'^2)}$$

$$x^2 y'^2 = k_1^2 (x + x y'^2)$$

$$x^2 y'^2 = x k_1^2 + x y'^2 k_1^2$$

$$x^2 y'^2 - x y'^2 k_1^2 = x(k_1^2)$$

$$\cancel{x} y_1^2 (x - k_1^2) = \cancel{x} k_1^2$$

$$y_1^2 = \frac{k_1^2}{x - k_1^2}$$

$$y_1 = \frac{k_1}{\sqrt{x - k_1^2}}$$

ie $\frac{dy}{dx} = \frac{k_1}{\sqrt{x - k_1^2}}$

$$\int dy = \int \frac{k_1}{\sqrt{x - k_1^2}}\, dx$$

$$y = 2k_1 \sqrt{x - k_1^2} + k_2$$

$$(y - k_2) = 2k_1 \sqrt{x - k_1^2}$$

Squaring

$$\boxed{(y - k_2)^2 = 4k_1^2 (x - k_1^2)}$$

This is equation of parabola which is the required geodesic.

Find the geodesics on a sphere of radius a.

OR.

Show that the shortest arc joining two points on a sphere is the minor arc of the great circle through them.

Solution:

Consider the elementary arc length ds in spherical polar coordinate system. (r, θ, ϕ) as co-ordinates of a point.

$$ds = \sqrt{(dr)^2 + (r\,d\theta)^2 + (r\sin\theta\, d\phi)^2}$$

here radius $r = a$ $\therefore dr = 0$

$$\Rightarrow \quad ds = \sqrt{(a\,d\theta)^2 + (a\sin\theta\, d\phi)^2}$$

$$ds = \sqrt{a^2 + a^2\sin^2\theta\,\frac{d\phi^2}{d\theta^2}}\cdot d\theta$$

$$ds = a\sqrt{1+\sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2}\,d\theta$$

Then arc length between two points $A(a, \theta_1, \phi_1)$ and $B(a, \theta_2, \phi_2)$ is given by

$$S = \int_A^B ds = a\int_{\theta_1}^{\theta_2}\sqrt{1+\sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2}\,d\theta.$$

Now we need to find the curve between two points A and B so that S is minimum.

ie we need to find geodesic between A & B.

$f = \sqrt{1+\sin^2\theta\ \phi'^2}$

f does not contain ϕ, $\therefore$ Euler's Equ^n is $\boxed{\dfrac{\partial f}{\partial \phi'} = k}$ —①

$$\frac{\partial f}{\partial \phi'} = \frac{\partial \sqrt{1+\sin^2\theta\ \phi'^2}}{\partial \phi'} = \frac{1}{\cancel{2}\sqrt{1+\sin^2\theta\ \phi'^2}} \times \sin^2\theta . \cancel{2}\phi'$$

$$\frac{\partial f}{\partial \phi'} = \frac{\sin^2\theta . \phi'}{\sqrt{1+\sin^2\theta\ \phi'^2}}$$

Sub in ①

$$\frac{\sin^2\theta . \phi'}{\sqrt{1+\sin^2\theta\ \phi'^2}} = k$$

$$\sin^2\theta . \phi' = k\sqrt{1+\sin^2\theta\ \phi'^2}$$

squaring

$$\sin^4\theta . (\phi')^2 = k^2 (1+\sin^2\theta . \phi'^2).$$

$$\sin^4\theta . (\phi')^2 - k^2 \sin^2\theta . \phi'^2 = k^2$$

$$\sin^2\theta (\sin^2\theta - k^2)\phi'^2 = k^2$$

$$\phi'^2 = \frac{k^2}{\sin^2\theta . (\sin^2\theta - k^2)} = \frac{k^2 \operatorname{cosec}^2\theta}{\sin^2\theta - k^2}$$

$$\phi' = \frac{k \operatorname{cosec}\theta}{\sqrt{\sin^2\theta - k^2}}.$$

$$\frac{d\phi}{d\theta} = \frac{k . \operatorname{cosec}\theta}{\sqrt{\sin^2\theta - k^2}}.$$

$$d\phi = \frac{k \operatorname{cosec}\theta}{\sqrt{\sin^2\theta - k^2}}\, d\theta$$

$$= \frac{k \operatorname{cosec}\theta}{\sin\theta\sqrt{1 - \frac{k^2}{\sin^2\theta}}}\, d\theta = \frac{k \operatorname{cosec}^2\theta}{\sqrt{1 - k^2\operatorname{cosec}^2\theta}}$$

$\frac{1}{\sin\theta} = \operatorname{cosec}\theta$, $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$.

$$d\phi = \frac{k \operatorname{cosec}^2\theta}{\sqrt{1 - k^2(1+\cot^2\theta)}}\, d\theta = \frac{k \operatorname{cosec}^2\theta}{\sqrt{(1-k^2) + k^2\cot^2\theta}}\, d\theta$$

Pats

$$\int d\phi = \int \frac{k \operatorname{cosec}^2\theta}{\sqrt{(1-k^2) + k^2\cot^2\theta}}\, d\theta$$

Put $k\cot\theta = t \Rightarrow k \operatorname{cosec}^2\theta \, d\theta = dt$

$$= \int \frac{dt}{\sqrt{(1-k^2) + t^2}}$$

$$\int \frac{1}{\sqrt{a+t^2}}\, dt = \cos^{-1}\left(\frac{t}{\sqrt{a}}\right)$$

$$\phi = \cos^{-1}\left(\frac{t}{\sqrt{1-k^2}}\right) + k_1$$

$$\boxed{\phi = \cos^{-1}\left(\frac{k\cot\theta}{\sqrt{1-k^2}}\right) + k_1.} \quad \text{---} (2)$$

This is the curve of minimum length between two points A & B on the sphere $x^2+y^2+z^2 = a^2$

Geodesic on sphere $x^2+y^2+z^2 = a^2$ →

Part II : To show that geodesic is part of the great circle on the sphere, $x^2+y^2+z^2=a^2$.

in polar co-ordinates $\boxed{x = a\sin\theta\cos\phi,\ y = a\sin\theta\sin\phi,\ z = a\cos\theta}$ — (*)

now consider $\phi = \cos^{-1}\left(\frac{k\cot\theta}{\sqrt{1-k^2}}\right) + k_1$

$\Rightarrow (\phi - k_1) = \cos^{-1}\left(\frac{k\cot\theta}{\sqrt{1-k^2}}\right)$

$$\cos(\phi - k_1) = \frac{k\cot\theta}{\sqrt{1-k^2}} = c.\cot\theta.$$

Put $\frac{k}{\sqrt{1-k^2}} = c$ | $\cos\phi\cos k_1 + \sin\phi\sin k_1 = c\cot\theta = c\frac{\cos\theta}{\sin\theta}$

trasfer c & $\sin\theta$ to L.H.S

$$\left(\frac{\cos k_1}{c}\right)\cos\phi\sin\theta + \left(\frac{\sin k_1}{c}\right)\sin\phi\sin\theta = \cos\theta$$

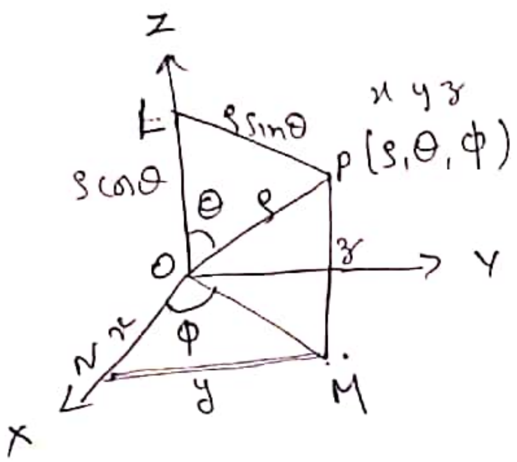
ie $\cos\theta = A\cos\phi\sin\theta + B\sin\phi\sin\theta.$

$\times^{ly}$ by a

$a\cos\theta = A(a\sin\theta\cos\phi) + B.(a\sin\theta\sin\phi)$

(*) $\Rightarrow$ $\boxed{z = Ax + By}$ which is eqn of plane passing through centre (0,0,0)

The geodesic is the intersection of this plane with sphere connecting two points on its surface — which is great circle.



In Δ^{le} PLO

$\sin\theta = \frac{LP}{\rho}$

$LP = \rho\sin\theta$

$\cos\theta = \frac{OL}{\rho}$

$OL = \rho\cos\theta.$

$\cancel{Lp} = OM = \rho\sin\theta$.

In Δ^{le} ONM

$\sin\phi = \frac{NM}{OM} = \frac{NM}{\rho\sin\theta}$

$y = NM = \rho\sin\theta\sin\phi$

$\cos\phi = \frac{ON}{OM} = \frac{ON}{\rho\sin\theta}$

$x = ON = \rho\sin\theta\cos\phi$

$z = OL = \rho\cos\theta$